Indian Statistical Institute, Bangalore
M. Math. First Year
Second Semester - Topology II

Final Exam Duration: 3 hours Date : April 19, 2016

Max Marks: 100

Remark: There are six questions here. Answer any five.

1. Let x be a vertex of a graph X. Let T be a spanning tree in X. Prove that $\pi_1(X, x)$ is isomorphic to the free group generated by the edges in $X \backslash T$ by exhibiting an explicit isomorphism between these two groups. You must show that your isomorphism is well defined and is indeed a group isomorphism. [20]

2. Present a five vertex triangulation of the Mobius band and explain why it triangulates the Mobius band. Using this triangulation, compute the fundamental group and the Betti numbers of the Mobius band. [20]

3. Let X be a simplicial complex and x be a vertex of X. Show that there is a well defined group homomorphism $f : \pi_1(X, x) \to H_1(X)$ given by $f([x_0, x_1, ... x_m]) = [x_0, x_1] + ... + [x_{m-1}, x_m]$ for any closed walk $x_0, x_1, ..., x_m$ at x. Show that f is onto, and its kernel is the commutator subgroup of $\pi_1(X, x)$. Hence conclude that $H_1(X)$ is the abelianization of $\pi_1(X, x)$. [20]

4. (a) Show that $H_0(X) \cong \pi_0(X)$ for any simplicial complex X.
   (b) Show that
$$\sum_{i=0}^{d}(-1)^i \beta_i = \sum_{i=0}^{d}(-1)^i f_i$$
   where $d = \dim(X)$ and $(\beta_0, ..., \beta_d)$ and $(f_0, ..., f_d)$ are the vectors of Betti numbers and of the face numbers of X. [20]

5. For any simplicial complex X and any face $\alpha \in X$, the link L_α of α in X is defined to be the sub-complex $L_\alpha = \{\beta \in X : \beta \cap \alpha = \varphi \text{ and } \beta \cup \alpha \in X\}$. X is said to be a homology sphere if, for every face α of X (including the empty face) the homology groups of L_α are isomorphic to the corresponding homology groups of the sphere of dimension $\dim(L_\alpha)$. (It can be shown that every triangulation of a sphere is a homology sphere.) Prove by induction on $\dim(X)$ that the face numbers of any homology sphere X of dimension d satisfy $f_i(X) \geq \begin{pmatrix} d+2 \\ i+1 \end{pmatrix}, 0 \leq i \leq d$, with equality only for the standard triangulation of S^d. [20]

6. Let (X, x) be a pointed topological space. Let P be the set of all maps $f : (I, 0) \to (X, x)$ when $I : [0, 1]$. Define the equivalence relation $\sim$ on P by $f_1 \sim f_2$ if $f_1(1) = f_2(1)$. Let $< f >$ be the $\sim$ class containing $f \in P$. For $f \in P$ and a neighborhood $\mho$ of $f(1)$, let $< \mho, < f >)$ be the set of all $< g >$ as g ranges over the continuations of f in $\mho$. Show that the family of such sets $< \mho, < f >)$ is the base for a path connected topology on $\widetilde{X} := \{< f >: f \in P\}$. [20]